



12.1 Invariant subspaces, eigenvalues and eigenvectors





Introduction

- In this topic, we will
 - Define invariant subspaces
 - Define eigenvalues and eigenvectors
 - Consider some consequences
 - Determine how to find eigenvalues
 - Determine how to find eigenvectors





Definitions

- Suppose that a linear operator $A: \mathcal{U} \rightarrow \mathcal{U}$ has property that it maps a vector onto a scalar multiple of itself
 - That is, there is a non-zero vector $\mathbf{u}$ such that $A\mathbf{u} = \lambda \mathbf{u}$
 - We will always exclude the zero vector from being such a candidate, as $A\mathbf{0} = \lambda \mathbf{0}$ for all scalars λ
- If there exists such a vector $\mathbf{u}$,
 - that vector is said to be an eigenvector of A , and the corresponding λ is the eigenvalue associated $\mathbf{u}$
 - If an $n \times n$ matrix has n eigenvalues (counting multiplicity), then the eigenvectors and their corresponding eigenvalues completely describe the behavior of the matrix





Not all matrices have real eigenvalues

- Unfortunately,
 - A real square matrix that has an odd dimension is only guaranteed to have at least one real eigenvalue
 - Also, a real square matrix that does not have at least one real eigenvalue must have at least complex conjugate pair of *eigenvalues*, which indicates a 2-dimensional invariant plane

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

- We will focus on finding conditions under which the eigenvectors and corresponding eigenvalues fully describe the behavior of the linear operator





Diagonal matrices

- For example, consider a matrix D is a diagonal $n \times n$ matrix, and $\mathbf{e}_k$ is the k^{th} canonical basis vector
 - In this case, $D\mathbf{e}_k = d_{k,k}\mathbf{e}_k$, so each canonical vector $\mathbf{e}_k$ is an eigenvector with a corresponding eigenvalue $d_{k,k}$

- For example, if $D = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -0.1 & 0 \\ 0 & 0 & 0 & 0.5 \end{pmatrix}$, we have that

$$\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -0.1 & 0 \\ 0 & 0 & 0 & 0.5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$D\mathbf{e}_1 = -3\mathbf{e}_1$$

$$D\mathbf{e}_2 = 2\mathbf{e}_2$$

$$D\mathbf{e}_3 = -0.1\mathbf{e}_3$$

$$D\mathbf{e}_4 = 0.5\mathbf{e}_4$$





Null spaces

- If a linear operator A has a non-trivial null space, then for each $\mathbf{u} \in \text{null}(A)$ has the property that $A\mathbf{u} = \mathbf{0} = 0\mathbf{u}$
 - That is, each vector in the null space is an eigenvector with a corresponding eigenvalue of $\lambda = 0$





Subspaces

- The null space is all eigenvectors that have an eigenvalue $\lambda = 0$
 - Consider all eigenvectors with the same eigenvalue λ

Theorem

Given a linear operator A , the collection of all eigenvectors for a given eigenvalue λ forms a subspace.

Proof:

If $\mathbf{u}_1$ and $\mathbf{u}_2$ are eigenvectors corresponding to an eigenvalue λ it follows that $A\mathbf{u}_1 = \lambda\mathbf{u}_1$ and $A\mathbf{u}_2 = \lambda\mathbf{u}_2$

$$\begin{aligned}\text{Thus, } A(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) &= \alpha A\mathbf{u}_1 + \beta A\mathbf{u}_2 \\ &= \alpha \lambda \mathbf{u}_1 + \beta \lambda \mathbf{u}_2 \\ &= \lambda(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2)\end{aligned}$$

Thus, $\alpha \mathbf{u}_1 + \beta \mathbf{u}_2$ is also an eigenvector with eigenvalue λ . ■





Eigenspace

- Thus, every eigenvalue is associated with an *eigenspace* of eigenvectors
 - Because it is a subspace, we can find a basis for that subspace
 - Where possible, we prefer that basis to be orthonormal





Linear independence of eigenvectors

Theorem

Suppose a linear operator A has m distinct eigenvalue $\lambda_1, \dots, \lambda_m$ and there are m eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_m$ corresponding to these eigenvalues, respectively; then the vectors are linearly independent.





Linear independence of eigenvectors

- Recall that if a collection of vectors is linearly dependent, there is a first vector $\mathbf{u}_\ell$ that must linearly depend on the $\ell - 1$ vectors where all $\ell - 1$ previous vectors are linearly independent
- We will proceed as follows: we will assume the converse of what the theorem states: that is, eigenvectors corresponding to different eigenvalues are linearly dependent
- If this is the case, there must be a first eigenvector that linearly depends on the previous $\ell - 1$ linearly independent vectors
- However, we will see this leads to a contradiction, as this will show that the previous $\ell - 1$ vectors are also linearly dependent
- Recall that m vectors being linearly independent is equivalent to saying that the only solution to

$$\alpha_1 \mathbf{u}_1 + \cdots + \alpha_m \mathbf{u}_m = \mathbf{0}$$

is the trivial solution $\alpha_1 = \cdots = \alpha_m = 0$





Linear independence of eigenvectors

Theorem

Suppose a linear operator A has m distinct eigenvalue $\lambda_1, \dots, \lambda_m$ and there are m eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_m$ corresponding to these eigenvalues, respectively; then the vectors are linearly independent.

Proof:

Assume that they are linearly dependent, in which case there is a first vector $\mathbf{u}_\ell$ that linearly depends on the previous $\ell - 1$ linearly independent vectors

$$\mathbf{u}_\ell = \alpha_1 \mathbf{u}_1 + \dots + \alpha_{\ell-1} \mathbf{u}_{\ell-1}$$

where not all α_k are not zero





Linear independence of eigenvectors

First, we note that $\mathbf{u}_\ell$ is an eigenvector, and thus $A\mathbf{u}_\ell = \lambda_\ell \mathbf{u}_\ell$

Consequently,

$$\begin{aligned} A\mathbf{u}_\ell &= \lambda_\ell(\alpha_1\mathbf{u}_1 + \cdots + \alpha_{\ell-1}\mathbf{u}_{\ell-1}) \\ &= \lambda_\ell\alpha_1\mathbf{u}_1 + \cdots + \lambda_\ell\alpha_{\ell-1}\mathbf{u}_{\ell-1} \end{aligned}$$

However, A is linear, so

$$\begin{aligned} A\mathbf{u}_\ell &= A(\alpha_1\mathbf{u}_1 + \cdots + \alpha_{\ell-1}\mathbf{u}_{\ell-1}) \\ &= \alpha_1 A\mathbf{u}_1 + \cdots + \alpha_{\ell-1} A\mathbf{u}_{\ell-1} \\ &= \alpha_1\lambda_1\mathbf{u}_1 + \cdots + \alpha_{\ell-1}\lambda_{\ell-1}\mathbf{u}_{\ell-1} \end{aligned}$$

Equating these two, we have

$$\lambda_\ell\alpha_1\mathbf{u}_1 + \cdots + \lambda_\ell\alpha_{\ell-1}\mathbf{u}_{\ell-1} = \alpha_1\lambda_1\mathbf{u}_1 + \cdots + \alpha_{\ell-1}\lambda_{\ell-1}\mathbf{u}_{\ell-1}$$





Linear independence of eigenvectors

Thus, given

$$\lambda_\ell \alpha_1 \mathbf{u}_1 + \cdots + \lambda_\ell \alpha_{\ell-1} \mathbf{u}_{\ell-1} = \alpha_1 \lambda_1 \mathbf{u}_1 + \cdots + \alpha_{\ell-1} \lambda_{\ell-1} \mathbf{u}_{\ell-1}$$

Bringing all the terms to one side, we have:

$$(\lambda_\ell - \lambda_1) \alpha_1 \mathbf{u}_1 + \cdots + (\lambda_\ell - \lambda_{\ell-1}) \alpha_{\ell-1} \mathbf{u}_{\ell-1} = \mathbf{0}$$

As these are different eigenvalues, none of these differences are zero, and by assumption, not all the α_k were zero

Thus, we have a linear combination of vectors from 1 to $\ell-1$ which equals the zero vector, but not all coefficients of which are zero

Therefore, the vectors $\mathbf{u}_1, \dots, \mathbf{u}_{\ell-1}$ are linearly dependent, which contradicts our assumption.

Therefore, eigenvectors corresponding to different eigenvalues are linearly independent. ■





Linear independence of eigenvectors

- As a consequence, a linear operator $A: \mathbf{F}^n \rightarrow \mathbf{F}^n$ cannot have more than n distinct eigenvalues





Eigenvalues of a self-adjoint operator

Theorem

If A is a self-adjoint linear operator,
its eigenvalues are real

Proof:

$$\begin{aligned} \text{First, } \langle \mathbf{u}, A\mathbf{u} \rangle &= \langle \mathbf{u}, \lambda\mathbf{u} \rangle & \text{but } \langle \mathbf{u}, A\mathbf{u} \rangle &= \langle A^*\mathbf{u}, \mathbf{u} \rangle \\ &= \lambda \langle \mathbf{u}, \mathbf{u} \rangle & &= \langle A\mathbf{u}, \mathbf{u} \rangle \\ & & &= \langle \lambda\mathbf{u}, \mathbf{u} \rangle \\ & & &= \lambda^* \langle \mathbf{u}, \mathbf{u} \rangle \end{aligned}$$

Thus, $\lambda = \lambda^*$, and thus λ is real. ■





Eigenvalues of a self-adjoint operator

Theorem

If A is a self-adjoint linear operator, eigenvectors corresponding to different eigenvalues are orthogonal.

Proof:

$$\begin{aligned} \text{First, } \langle \mathbf{u}_1, A\mathbf{u}_2 \rangle &= \langle \mathbf{u}_1, \lambda_2 \mathbf{u}_2 \rangle \quad \text{but } \langle \mathbf{u}_1, A\mathbf{u}_2 \rangle = \langle A^* \mathbf{u}_1, \mathbf{u}_2 \rangle \\ &= \lambda_2 \langle \mathbf{u}_1, \mathbf{u}_2 \rangle &= \langle A\mathbf{u}_1, \mathbf{u}_2 \rangle \\ & &= \langle \lambda_1 \mathbf{u}_1, \mathbf{u}_2 \rangle \\ & &= \lambda_1 \langle \mathbf{u}_1, \mathbf{u}_2 \rangle \end{aligned}$$

Thus, as $\lambda_1 \neq \lambda_2$, and thus $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$. ■





Invertibility and zero eigenvalues

Theorem

A linear operator $A: \mathbf{F}^n \rightarrow \mathbf{F}^n$ operator is invertible if and only if it has no zero eigenvalues.

Proof:

Assume A is invertible.

In this case, the null space is just the zero vector.

Thus, no non-zero vector maps to the zero vector.

Thus, no vector has the property $A\mathbf{u} = \mathbf{0} = 0\mathbf{u}$, so no vector is an eigenvector with eigenvalue 0.

Assume A is not invertible.

In this case, the null space has at least one non-zero vector $\mathbf{u}$.

Thus, $A\mathbf{u} = \mathbf{0} = 0\mathbf{u}$, and thus, A has a zero eigenvalue. ■





Invertibility and zero eigenvalues

Theorem

A linear operator $A: \mathbf{F}^n \rightarrow \mathbf{F}^n$ operator is invertible if and only if it has no zero eigenvalues.

Proof:

Assume A is invertible.

In this case, the null space is just the zero vector.

Thus, no non-zero vector maps to the zero vector.

Thus, no vector has the property $A\mathbf{u} = \mathbf{0} = 0\mathbf{u}$, so no vector is an eigenvector with eigenvalue 0.

Assume A is not invertible.

In this case, the null space has at least one non-zero vector $\mathbf{u}$.

Thus, $A\mathbf{u} = \mathbf{0} = 0\mathbf{u}$, and thus, A has a zero eigenvalue. ■





Finding eigenvalues

- In order to find eigenvalues, we must recall first that a linear operator $A: \mathbf{F}^n \rightarrow \mathbf{F}^n$ is invertible if and only if the determinant is not zero.
 - Alternatively, such a linear operator is non-invertible if and only if the determinant is zero.
- Recall also that a linear operator $A: \mathbf{F}^n \rightarrow \mathbf{F}^n$ is invertible if and only if the null space is just the zero vector.
 - Alternatively, such a linear operator is non-invertible if and only if the null space has dimension of at least one.
- Finally, we just saw that a linear operator $A: \mathbf{F}^n \rightarrow \mathbf{F}^n$ is invertible if and only that operator has no zero eigenvalues.
 - Alternatively, such a linear operator is non-invertible if and only if the operator has at least one zero eigenvalue.





Finding eigenvalues

- Now, suppose that we want to find values of λ such that

$$A\mathbf{u} = \lambda\mathbf{u}$$

- This is equivalent to saying that

$$\lambda\mathbf{u} - A\mathbf{u} = \mathbf{0}$$

- Recall that $I\mathbf{u} = \mathbf{u}$, so we can substitute this into the equation:

$$\lambda I\mathbf{u} - A\mathbf{u} = \mathbf{0}$$

- Now this is of the form $B\mathbf{u} - A\mathbf{u} = (B - A)\mathbf{u}$, so

$$(\lambda I - A)\mathbf{u} = \mathbf{0}$$

- But $\lambda I - A$ is a linear operator, so for this to be possible, the null space of $\lambda I - A$ must have dimension greater than or equal to one
 - Therefore, we must find $\lambda I - A$ such that $\lambda I - A$ is not invertible
 - Therefore, we could find $\lambda I - A$ such that $\det(\lambda I - A) = 0$





Characteristic polynomial

- The formula $\det(\lambda I - A)$ is actually
a polynomial in λ of degree n
 - It will therefore have n complex roots, counting multiplicity
 - If the matrix was real, the coefficients will be real
 - The n roots will either be real or come in complex conjugate pairs
- This polynomial is called
the *characteristic polynomial* of the linear operator A





Finding eigenvalues

- Let's do this: $A_1 = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$
 - Thus, $\lambda I - A_1 = \begin{pmatrix} \lambda - 2 & -3 \\ 0 & \lambda - 4 \end{pmatrix}$
$$\det(\lambda I - A_1) = (\lambda - 2)(\lambda - 4) - 0 \cdot (-3)$$
$$= (\lambda - 2)(\lambda - 4)$$
 - Thus, two eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 4$





Finding eigenvalues

- Let's do this: $A_2 = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$
 - Thus, $\lambda I - A_2 = \begin{pmatrix} \lambda - 2 & -3 \\ -1 & \lambda - 4 \end{pmatrix}$
$$\begin{aligned} \det(\lambda I - A_2) &= (\lambda - 2)(\lambda - 4) - (-1)(-3) \\ &= \lambda^2 - 6\lambda + 8 - 3 \\ &= \lambda^2 - 6\lambda + 5 \end{aligned}$$
 - Thus, two eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 5$





Finding eigenvalues

- Let's do this: $A_3 = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$
 - Thus, $\lambda I - A_3 = \begin{pmatrix} \lambda - 2 & -3 \\ -3 & \lambda - 2 \end{pmatrix}$
$$\begin{aligned} \det(\lambda I - A_3) &= (\lambda - 2)(\lambda - 2) - (-3)(-3) \\ &= \lambda^2 - 4\lambda + 4 - 9 \\ &= \lambda^2 - 4\lambda - 5 \end{aligned}$$
 - Thus, two eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 5$





Finding eigenvalues

- Let's do this: $A_4 = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$
 - Thus, $\lambda I - A_4 = \begin{pmatrix} \lambda - 6 & 3 \\ 2 & \lambda - 1 \end{pmatrix}$ and therefore
$$\begin{aligned} \det(\lambda I - A_4) &= (\lambda - 6)(\lambda - 1) - 2 \cdot 3 \\ &= \lambda^2 - 7\lambda + 6 - 6 \\ &= \lambda^2 - 7\lambda \end{aligned}$$
 - Thus, two eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 7$



Finding eigenvalues

- Let's do this: $A_5 = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$

- Thus, $\lambda I - A_5 = \begin{pmatrix} \lambda - 2 & 3 \\ -3 & \lambda - 2 \end{pmatrix}$

$$\begin{aligned} \det(\lambda I - A_5) &= (\lambda - 2)(\lambda - 2) - 3 \cdot (-3) \\ &= \lambda^2 - 4\lambda + 4 + 9 \\ &= \lambda^2 - 4\lambda + 13 \end{aligned}$$

- Thus, two roots are $\frac{4 \pm \sqrt{16 - 4 \cdot 13}}{2} = \frac{4 \pm \sqrt{-36}}{2}$
 $= 2 \pm 3j$





Finding eigenvalues

- Let's do this: $A_6 = \begin{pmatrix} 0 & -5 \\ 5 & 0 \end{pmatrix}$
 - Thus, $\lambda I - A_6 = \begin{pmatrix} \lambda - 0 & 5 \\ -5 & \lambda - 0 \end{pmatrix}$ and therefore
$$\begin{aligned} \det(\lambda I - A_6) &= \lambda^2 - 5 \cdot (-5) \\ &= \lambda^2 + 25 \end{aligned}$$
 - Thus, two roots are $\pm 5j$





Finding eigenvalues

- Let's do this: $B_1 = \begin{pmatrix} 2 & 2 & 10 \\ 2 & 4 & 4 \\ 2 & 1 & 1 \end{pmatrix}$

– Thus, $\lambda I - B_1 = \begin{pmatrix} \lambda - 2 & -2 & -10 \\ -2 & \lambda - 4 & -4 \\ -2 & -1 & \lambda - 1 \end{pmatrix}$

$$\begin{aligned} \det(\lambda I - B_1) &= (\lambda - 2)(\lambda - 4)(\lambda - 1) + (-2)(-4)(-2) + (-10)(-2)(-1) \\ &\quad - (\lambda - 2)(-4)(-1) - (-2)(-2)(\lambda - 1) - (-10)(\lambda - 4)(-2) \\ &= \lambda^3 - 7\lambda^2 + 14\lambda - 8 - 16 - 20 - 4\lambda + 8 - 4\lambda + 4 - 20\lambda + 80 \\ &= \lambda^3 - 7\lambda^2 - 14\lambda + 48 \\ &= (\lambda - 8)(\lambda + 3)(\lambda - 2) \end{aligned}$$

– Thus, three eigenvalues are $\lambda_1 = -3$, $\lambda_2 = 2$ and $\lambda_3 = 8$





Finding eigenvalues

- Let's do this: $B_2 = \begin{pmatrix} 1 & 1 & 7 \\ 1 & -3 & 2 \\ -2 & 10 & 1 \end{pmatrix}$

- Thus, $\lambda I - B_2 = \begin{pmatrix} \lambda - 1 & -1 & -7 \\ -1 & \lambda + 3 & -2 \\ 2 & -10 & \lambda - 1 \end{pmatrix}$

$$\begin{aligned} \det(\lambda I - B_2) &= (\lambda - 1)(\lambda + 3)(\lambda - 1) + (-1)(-2)(2) + (-7)(-1)(-10) \\ &\quad - (\lambda - 1)(-2)(-10) - (-1)(-1)(\lambda - 1) - (-7)(\lambda + 3)(2) \\ &= \lambda^3 + \lambda^2 - 5\lambda + 3 + 4 - 70 - 20\lambda + 20 - \lambda + 1 + 14\lambda + 42 \\ &= \lambda^3 + \lambda^2 - 12\lambda \\ &= \lambda(\lambda + 4)(\lambda - 3) \end{aligned}$$

- Thus, three eigenvalues are $\lambda_1 = -4$, $\lambda_2 = 0$ and $\lambda_3 = 3$





Finding eigenvalues

- Let's do this: $B_3 = \begin{pmatrix} -2 & 1 & 7 \\ 3 & 1 & -1 \\ 3 & 9 & 6 \end{pmatrix}$

- Thus, $\lambda I - B_3 = \begin{pmatrix} \lambda + 2 & -1 & -7 \\ -3 & \lambda - 1 & 1 \\ -3 & -9 & \lambda - 6 \end{pmatrix}$

$$\begin{aligned} \det(\lambda I - B_3) &= (\lambda + 2)(\lambda - 1)(\lambda - 6) + (-1)(1)(-3) + (-7)(-3)(-9) \\ &\quad - (\lambda + 2)(1)(-9) - (-1)(-3)(\lambda - 6) - (-7)(\lambda - 1)(-3) \\ &= \lambda^3 - 5\lambda^2 - 8\lambda + 12 + 3 - 189 + 9\lambda + 18 - 3\lambda + 18 - 21\lambda + 21 \\ &= \lambda^3 - 5\lambda^2 - 23\lambda - 117 \\ &= (\lambda - 9)(\lambda^2 + 4\lambda + 13) \end{aligned}$$

- Thus, one eigenvalue is $\lambda_1 = 9$ with other roots $-2 \pm 3j$





Finding eigenvectors

- Now, suppose we have an eigenvalue λ such that

$$A\mathbf{u} = \lambda\mathbf{u}$$

for some $\mathbf{u}$

- How do we find $\mathbf{u}$?
- How about solving the homogenous system of linear equations?

$$(\lambda I - A)\mathbf{u} = \mathbf{0}$$

- Given an eigenvalue λ ,
we are finding a basis for the null space of $\lambda I - A$
- The number of linearly independent eigenvectors for a given eigenvalue
 - Will be at least 1
 - Shall not exceed the multiplicity of the eigenvalue





Finding eigenvectors

- Recall that $A_1 = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$ had eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 4$
- We will solve $\left(\begin{array}{cc|c} \lambda - 2 & -3 & 0 \\ 0 & \lambda - 4 & 0 \end{array} \right)$ substituting each value of λ

– For $\lambda_1 = 2$, $\left(\begin{array}{cc|c} 2-2 & -3 & 0 \\ 0 & 2-4 & 0 \end{array} \right) = \left(\begin{array}{cc|c} 0 & -3 & 0 \\ 0 & -2 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 0 & -3 & 0 \\ 0 & 0 & 0 \end{array} \right)$

- Thus, $\mathbf{u}_1 = \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

– For $\lambda_2 = 4$, $\left(\begin{array}{cc|c} 4-2 & -3 & 0 \\ 0 & 4-4 & 0 \end{array} \right) = \left(\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array} \right)$

- Thus, $\mathbf{u}_2 = \begin{pmatrix} 1.5\alpha_2 \\ \alpha_2 \end{pmatrix} = \alpha_2 \begin{pmatrix} 1.5 \\ 1 \end{pmatrix}$





Finding eigenvectors

- Recall that $A_2 = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ had eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 5$
- We will solve $\left(\begin{array}{cc|c} \lambda-2 & -3 & 0 \\ -1 & \lambda-4 & 0 \end{array} \right)$ substituting each value of λ

– For $\lambda_1 = 1$, $\left(\begin{array}{cc|c} 1-2 & -3 & 0 \\ -1 & 1-4 & 0 \end{array} \right) = \left(\begin{array}{cc|c} -1 & -3 & 0 \\ -1 & -3 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} -1 & -3 & 0 \\ 0 & 0 & 0 \end{array} \right)$

- Thus, $\mathbf{u}_1 = \begin{pmatrix} -3\alpha_2 \\ \alpha_2 \end{pmatrix} = \alpha_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

– For $\lambda_2 = 5$, $\left(\begin{array}{cc|c} 5-2 & -3 & 0 \\ -1 & 5-4 & 0 \end{array} \right) = \left(\begin{array}{cc|c} 3 & -3 & 0 \\ -1 & 1 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 3 & -3 & 0 \\ 0 & 0 & 0 \end{array} \right)$

- Thus, $\mathbf{u}_2 = \begin{pmatrix} \alpha_2 \\ \alpha_2 \end{pmatrix} = \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$





Finding eigenvectors

- Recall that $A_4 = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$ had eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 7$
- We will solve $\left(\begin{array}{cc|c} \lambda - 6 & 3 & 0 \\ 2 & \lambda - 1 & 0 \end{array} \right)$ substituting each value of λ

– For $\lambda_1 = 0$, $\left(\begin{array}{cc|c} 0 - 6 & 3 & 0 \\ 2 & 0 - 1 & 0 \end{array} \right) = \left(\begin{array}{cc|c} -6 & 3 & 0 \\ 2 & -1 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} -6 & 3 & 0 \\ 0 & 0 & 0 \end{array} \right)$

- Thus, $\mathbf{u}_1 = \begin{pmatrix} 0.5\alpha_2 \\ \alpha_2 \end{pmatrix} = \alpha_2 \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$

– For $\lambda_2 = 7$, $\left(\begin{array}{cc|c} 7 - 6 & 3 & 0 \\ 2 & 7 - 1 & 0 \end{array} \right) = \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array} \right)$

- Thus, $\mathbf{u}_2 = \begin{pmatrix} -3\alpha_2 \\ \alpha_2 \end{pmatrix} = \alpha_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix}$





Finding eigenvalues

- Recall $B_1 = \begin{pmatrix} 2 & 2 & 10 \\ 2 & 4 & 4 \\ 2 & 1 & 1 \end{pmatrix}$ had eigenvalues $\lambda_1 = -3$, $\lambda_2 = 2$ and $\lambda_3 = 8$
 $\lambda I - B_1 = \begin{pmatrix} \lambda - 2 & -2 & -10 \\ -2 & \lambda - 4 & -4 \\ -2 & -1 & \lambda - 1 \end{pmatrix}$

- For $\lambda_1 = -3$, $\begin{pmatrix} -5 & -2 & -10 \\ -2 & -7 & -4 \\ -2 & -1 & -4 \end{pmatrix} \sim \begin{pmatrix} -5 & -2 & -10 \\ 0 & -6.2 & 0 \\ 0 & -0.2 & 0 \end{pmatrix} \sim \begin{pmatrix} -5 & -2 & -10 \\ 0 & -6.2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- For $\lambda_2 = 2$, $\begin{pmatrix} 0 & -2 & -10 \\ -2 & -2 & -4 \\ -2 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} -2 & -2 & -4 \\ 0 & -2 & -10 \\ 0 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} -2 & -2 & -4 \\ 0 & -2 & -10 \\ 0 & 0 & 0 \end{pmatrix}$

- For $\lambda_3 = 8$, $\begin{pmatrix} 6 & -2 & -10 \\ -2 & 4 & -4 \\ -2 & -1 & 7 \end{pmatrix} \sim \begin{pmatrix} 6 & -2 & -10 \\ 0 & 3\frac{1}{3} & -7\frac{1}{3} \\ 0 & -1\frac{2}{3} & 3\frac{2}{3} \end{pmatrix} \sim \begin{pmatrix} 6 & -2 & -10 \\ 0 & 3\frac{1}{3} & -7\frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix}$

$$\mathbf{u}_1 = \begin{pmatrix} -2\alpha_3 \\ 0 \\ \alpha_3 \end{pmatrix} = \alpha_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix} 3\alpha_3 \\ -5\alpha_3 \\ \alpha_3 \end{pmatrix} = \alpha_3 \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} 2.4\alpha_3 \\ 2.2\alpha_3 \\ \alpha_3 \end{pmatrix} = \alpha_3 \begin{pmatrix} 2.4 \\ 2.2 \\ 1 \end{pmatrix}$$





Summary

- Following this topic, you now
 - Understand the definition of an eigenvalue and eigenvector
 - Know that eigenvectors corresponding to different eigenvalues are linearly independent
 - Know that the eigenvalues of self-adjoint operators are real and that eigenvectors corresponding to different eigenvalues are orthogonal
 - Know how to find eigenvalues
 - Know how to find eigenvectors





References

- [1] https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

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